Knot theory, knot practice – Problems 2

LMS Summer School, Bath, July 2025 **Ailsa Keating**

N.B. There are more questions than your group will be about to talk about in an hour! The idea is to have a choice, depending on which topics you found most interesting so far. You're welcome to think about the left-over questions at a later time (but please don't feel obliged to). Questions marked ($\star$) are more exploratory.

1. Sketch a proof that the product operation on oriented knots is associative and commutative.

2. Give some examples to show that the product operation is not well-defined for unoriented knots.

3. Sketch a proof that the unknot is has no non-trivial factors (without assuming the factorisation theorem from lectures). ($\star$) Do you think that all non-trivial torus knots are prime?

4. ($\star$) A Brunnian link is a non-trivial link L such that removing any of the components of L gives the trivial link.

   (a) Find an n-component Brunnian link for every integer $n \geq 2$.

   (b) Do you think all Brunnian links are prime? (We say that a link is prime if it is non-trivial, non-split, and has no proper factors.)

5. For $p > q \geq 2$, show that the torus link $T(p, q)$ has a Seifert surface of genus $\frac{1}{2}(p-1)(q-1)$.

   See below for a definition of genus. Fact: This minimises the genus of orientable surfaces in $\mathbb{R}^3$ with boundary $T(p, q)$! This is proved using an invariant called the Alexander polynomial.

6. Find an example which shows that for a given a link diagram, changing the orientation of a component can change the genus of the Seifert surface.

7. Draw Seifert surfaces for pretzel knots. What genera can they have? *[You may want to organise cases depending on the parities of p, q and r.]*

8. Suppose that L is a link with an odd number of components. Prove that the normalised bracket polynomial $\widetilde{V}_L(A)$ is a polynomial in A^4.

   ($\star$) What about links with an even number of components?

9. Compute the normalised bracket polynomial of the torus knot $T(2, p)$.

10. Show that the normalised bracket polynomial has the following properties:

    (a) $\widetilde{V}_L = \widetilde{V}_{-L}$

    (b) $\widetilde{V}_{L^*}(A) = \widetilde{V}_L(A^{-1})$

    (c) If a link can be factorised as $L_1 \# L_2$, then

    $$\widetilde{V}_{L_1 \# L_2} = \widetilde{V}_{L_1} \widetilde{V}_{L_2}$$

    (d) ($\star$) For a split link $L_1 \sqcup L_2$, we have

    $$\widetilde{V}_{L_1 \sqcup L_2}(A) = (-A^{-2} - A^2) \widetilde{V}_{L_1}(A) \widetilde{V}_{L_2}(A)$$

Note: If you haven't met genus yet, the genus of the Seifert surface F for a diagram D can be defined as $g(F) = \frac{1}{2}(2 - v(D) + c(D) - n(D))$, where $c(D)$ is the number of crossings of D; $v(D)$ is the number of embedded circles we obtained after resolving crossings in Seifert's algorithm; and $n(D)$ is the number of components of the link whose diagram is D.

v2 updated 15/07/2025. Comments, corrections or suggestions are all welcome! You can either talk to me in person or write to `amk50@cam.ac.uk`.